Fitted Modified Upwind Difference Scheme for Solving Singularly Perturbed Differential- Difference Equations with negative shift

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Abstract

In this paper an exponentially fitted modified upwind difference scheme is presented for solving differential-difference equations of convection-diffusion type with small delay parameter δ of o(ε), whose solution exhibits boundary layer behavior. First, the singularly perturbed differential-difference equation is replaced by an asymptotically equivalent singular perturbation problem. Then, fitted modified upwind scheme is developed and a three term recurrence relation is obtained. The resulted tri-diagonal system is solved by Discrete Invariant Imbedding Algorithm. The method is demonstrated by implementing on several model examples by taking various values for the delay parameter and the perturbation parameter.

Key words: *Singular perturbation problems, Differential-difference equations, Boundary layer, Delay parameter, finite differences*

Introduction

Differential-difference equations, arise in mathematical modeling of various practical phenomenon for example in hydrodynamics of liquid helium[6], diffusion in polymers[12], micro-scale heat transfer[18], a variety of model for physiological processes or diseases etc. Recently many researchers have started developing numerical methods for solving these problems. Chakravarthy and Rao[15] presented Numerov method for solving singularly perturbed delay differential equations containing both negative and positive shifts.. Kumar and Kadalbajoo [7] presented a parameter uniform numerical method for similar problems. Lange and Miura[9, 10] have presented various numerical methods to solve this type of problems. Moreover, numerical analysis of the problem under consideration have been done by Pratima and Sharma [16].

In this paper, fitted modified upwind finite difference scheme is presented for solving a singularly perturbed delay differential equation with layer behavior. First, the singularly perturbed delay differential equation is replaced by an asymptotically equivalent second order singularly perturbed two point boundary value problem. Then, a fitting factor is introduced into modified upwind finite difference scheme and a three term recurrence relation is obtained. Thomas algorithm is used to solve the tri-diagonal system and the stability of the algorithm is also considered. To show the applicability of the method, we have applied it on several numerical examples by taking different values for perturbation and delay parameters.

2. Description of the Method

Consider the singularly perturbed differential-difference equation of the form:

$$\varepsilon y'' + a(x)y'(x-\delta) + b(x)y(x) = f(x), \qquad 0 < x < 1 \tag{1}$$

under the interval and boundary conditions

$$y(0) = \alpha, \quad \text{and} \quad y(1) = \beta \tag{2}$$

where $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions, $b(x) \le 0$, $0<\varepsilon<<1$ and δ =o(ε) is the delay parameter such that $(\varepsilon - \delta a(x)) > 0$ for all $x \in [0,1]$. Furthermore, α and β are positive constants. When δ is zero equation (1) reduces to a singularly perturbed ordinary differential equation which with small ε exhibits layers and turning points depending upon the coefficient of convection term. The layer is maintained for $\delta \ne 0$ but sufficiently small.

2.1. Left End Boundary Layer Problem

We assume that $a(x) \ge M > 0$ throughout the interval [0,1], for some positive constant M and also $\varepsilon - \delta a(x) > 0, \forall x \in [0,1]$. This assumption implies that the boundary layer will be at the left end, that is in the neighborhood of x =0.

Taking the Taylor series expansion of the term $y'(x-\delta)$ around x we have

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \qquad (3)$$

Substituting (3) into (1) we get an asymptotically equivalent two point boundary value problem:

$$\varepsilon' y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \qquad (4)$$

with $$y(0) = \alpha, \quad \mathrm{y}(1) = \beta \qquad (5)$$

where $\varepsilon' = \varepsilon - \delta a(x)$

The transformation from (1) to (4) is admitted because of the condition that δ is sufficiently small. For further discussion on the validity of the transition one can refer El'sgolts & Norkin[5].

From the theory of singular perturbations, it is known that the solution of (4)-(5) is of the form [cf. O' Malley[11] pp. 22-26]

$$y(x) = y_0(x) + \frac{a(0)}{a(x)}(\alpha - y_0(0))e^{-\int_0^x \left(\frac{a(x)}{\varepsilon'} - \frac{b(x)}{a(x)}\right)dx} + o(\varepsilon') \qquad (6)$$

where $y_0(x)$ is the solution of the reduced problem

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), y_0(1) = \beta$$

By taking Taylor series expansion for $a(x)$ and $b(x)$ about the point '0' and restricting to their first terms, (6) becomes

$$y(x) = y_0(x) + (\alpha - y_0(0))e^{-\int_0^x \left(\frac{a(0)}{\varepsilon'} - \frac{b(0)}{a(0)}\right)dx} + o(\varepsilon') \qquad (7)$$

Now we divide the interval [0,1] into N equal subintervals of mesh size $h = \frac{1}{N}$ so that

$x_i = ih, \quad i = 0,1,2...N$.

From (7) we have

$$y(x_i) = y_0(x_i) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon'} - \frac{b(0)}{a(0)}\right)x_i} + o(\varepsilon')$$

i.e., $$y(ih) = y_0(ih) + (\alpha - y_0(0))e^{-\left(\frac{a(0)}{\varepsilon'} - \frac{b(0)}{a(0)}\right)ih} + o(\varepsilon')$$

Therefore

$$\lim_{h \to 0} y(ih) = y_0(0) + (\alpha - y_0(0))e^{-\left(\frac{a^2(0) - \varepsilon' b(0)}{a(0)}\right)i\rho} + o(\varepsilon') \qquad (8)$$

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where $\rho = h/\varepsilon'$

By Taylor's series expansion

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_i)}{h} - \frac{h}{2} y''(x_i) + O(h^2) \tag{9}$$

By substituting (9) in (4), we get the modified upwind finite difference scheme

$$\left(\varepsilon' - \frac{h}{2} a_i\right)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a_i\left(\frac{y(x_{i+1}) - y(x_i)}{h}\right) + b_i y_i = f_i + O(h^2), i = 1,2,3,\cdots, \mathrm{N}-1 \tag{10}$$

where $a(x_i) = a_i$; $b(x_i) = b_i$; $y(x_i) = y_i$, $f(x_i) = f_i$;

Now introducing a fitting factor σ into (10) we get

$$\sigma\left(\varepsilon' - \frac{h}{2} a_i\right)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a_i\left(\frac{y(x_{i+1}) - y(x_i)}{h}\right) + b_i y_i = f_i + O(h^2), i = 1,2,3,\cdots, \mathrm{N}-1 \tag{11}$$

with $y_0 = \alpha$ and $y_N = \beta$. (12)

Multiplying by h and taking the limit as $h \to 0$ we get

$$\lim_{h\to 0}\left[\sigma\left(\frac{1}{\rho} - \frac{a(ih)}{2}\right)\left(y_{i+1} - 2y_i + y_{i-1}\right) + a(ih)\left(y_{i+1} - y_i\right)\right] = 0$$

if $f(x_i) - b(x_i)y_i$ is bounded.

Therefore,

$$\lim_{h\to 0}\left[\sigma\left(\frac{1}{\rho} - \frac{a(ih)}{2}\right)\left(y(ih+h) - 2y(ih) + y(ih-h)\right) + a(ih)\left(y(ih+h) - y(ih)\right)\right] = 0 \tag{13}$$

substituting (8) in (13) and simplifying, we get the value of fitting factor as:

$$\sigma = \frac{\rho a(0)}{4}\left(\frac{2\rho a(0)}{2 - \rho a(0)}\right)\left[\frac{(1 - e^{-\left(\frac{a^{2}(0) - \varepsilon' b(0)}{a(0)}\right)\rho}}{Sinh^2\left[\left(\frac{a^2(0) - \varepsilon' b(0)}{2a(0)}\right)\rho\right]}\right] \tag{14}$$

From (9) we have

$$\left[\frac{\sigma}{h^2}\left(\varepsilon' - \frac{h}{2}a_i\right)\right]y_{i-1} - \left[\frac{2\sigma}{h^2}\left(\varepsilon' - \frac{h}{2}a_i\right) + \frac{a_i}{h} - b\right]y_i + \left(\frac{\sigma}{h^2}\left(\varepsilon' - \frac{h}{2}a_i\right) + \frac{a_i}{h}\right)y_{i+1} = f_i \tag{15}$$

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where and $f_i = f(x_i)$ considered for convenience.

The above equation can also be written as the three term recurrence relation

$$E_i y_{i-1} - F_i y_i + G y_{i+1} = H_i \tag{16}$$

where

$$E_i = \frac{\sigma}{h^2}\left(\varepsilon' - \frac{h}{2}a_i\right) \tag{17}$$

$$F_i = \frac{2\sigma}{h^2}\left(\varepsilon' - \frac{h}{2}a_i\right) + \frac{a_i}{h} - b_i \tag{18}$$

$$G_i = \frac{\sigma}{h^2}\left(\varepsilon' - \frac{h}{2}a_i\right) + \frac{a_i}{h} \tag{19}$$

$$H_i = f_i \tag{20}$$

This three term recurrence relation can be solved by Discrete-Invariant Imbedding Algorithm.

2.2. Right End Boundary Layer Problem

Now we consider (4)-(5) and assume that $a(x) \le M < 0$ throughout the interval [0,1], where M is a negative constant. This assumption implies that the boundary layer will be in the neighborhood of $x = 1$.

Thus, from the theory of singular perturbations, it is known that the solution of (4)-(5) is of the form [cf.O' Malley : 22-26]

$$y(x) = y_0(x) + \frac{a(1)}{a(x)}(\beta - y_0(1))e^{\int_x^1 \left(\frac{a(x)}{\varepsilon'} - \frac{b(x)}{a(x)}\right)dx} + o(\varepsilon') \tag{21}$$

where $y_0(x)$ is the solution of the reduced problem

$$a(x)y_0'(x) + b(x)y_0(x) = f(x), \qquad y_0(0) = \alpha \tag{22}$$

By taking Taylor series expansion for $a(x)$ and $b(x)$ about the point '1' and restricting to their first terms, (21) becomes

$$y(x) = y_0(x) + (\beta - y_0(1))e^{\int_x^1 \left(\frac{a(1)}{\varepsilon'} - \frac{b(1)}{a(1)}\right)dx} + o(\varepsilon') \tag{23}$$

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Now we divide the interval [0,1] into N equal subintervals of mesh size $h = \frac{1}{N}$ so that $x_i = ih, i = 0,1,2...N$. From (21) we have

$$y(x_i) = y_0(x_i) + (\beta - y_0(1))e^{\left(\frac{a(1)}{\varepsilon} - \frac{b(1)}{a(1)}\right)(1-x_i)} + o(\varepsilon') \tag{24}$$

$$\text{i.e.,}\; y(ih) = y_0(ih) + (\beta - y_0(1))e^{\left(\frac{a(1)}{e'} - \frac{b(1)}{a(1)}\right)(1-ih)} + o(\varepsilon') \tag{25}$$

Therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1))e^{-\left(\frac{a^2(1) - \varepsilon' b(1)}{a(1)}\right)\left(\frac{1}{\varepsilon'} - i\rho\right)} + o(\varepsilon') \tag{26}$$

where $\rho = h/\varepsilon'$

Now we consider the second order modified upwind finite difference scheme in (4)

$$\varepsilon'\sigma\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + a(x_i)\left(\frac{y_i - y_{i-1}}{h}\right) + b(x_i)y(x_i) = f(x_i);\; 1 \le i \le N-1 \tag{27}$$

$y_0 = \alpha, y_N = \beta,$ where σ is a fitting factor which is to be determined in such a way that the solution of (27) converges uniformly to the solution of (1)-(2). Multiplying (27) by h and taking the limits as $h \to 0$, we get

$$\lim_{h\to 0}\left[\frac{\sigma}{\rho}\big(y(ih+h) - 2y(ih) + y(ih-h)\big) + a(ih)\big(y(ih) - y(ih-h)\big)\right] = 0 \tag{28}$$

if $f(x_i) - b(x_i)y_i$ is bounded.

Substituting (26) in (28) and simplifying, we get the value of fitting factor as:

$$\sigma = \frac{2\rho a(0)}{2 + \rho a(0)}\left[\frac{\left(e^{-\left(\frac{a^2(1) - \varepsilon' b(1)}{a(1)}\rho\right)} - 1\right)}{Sinh^2\left(\left(\frac{a^2(1) - \varepsilon' b(1)}{a(1)}\right)\frac{\rho}{2}\right)}\right] \tag{29}$$

From (27) we have

$$\left(\frac{\sigma}{h^2}\left(\varepsilon' + \frac{h}{2}a_i\right) - \frac{a_i}{h}\right)y_{i-1} - \left(\frac{2\sigma}{h^2}\left(\varepsilon' + \frac{h}{2}a_i\right) - \frac{a_i}{h} - b_i\right)y_i + \left(\frac{\sigma}{h^2}\left(\varepsilon' + \frac{h}{2}a_i\right)\right)y_{i+1} = f_i \tag{30}$$

where $a_i = a(x_i), b_i = b(x_i)$ and $f_i = f(x_i)$ are considered for convenience.

The equation (30) can also be written as

$$E_i y_{i-1} - F_i y_i + G y_{i+1} = H_i \tag{31}$$

where

$$E_i = \frac{\sigma}{h^2}\left(\varepsilon' + \frac{h}{2}a_i\right) - \frac{a_i}{h} \tag{32}$$

$$F_i = \frac{2\sigma}{h^2}\left(\varepsilon' + \frac{h}{2}a_i\right) - \frac{a_i}{h} - b_i \tag{33}$$

$$G_i = \frac{\sigma}{h^2}\left(\varepsilon' + \frac{h}{2}a_i\right) \tag{34}$$

$$H_i = f_i \tag{35}$$

This three term recurrence relation can be easily solved by Discrete-Invariant Imbedding Algorithm.

3. Thomas Algorithm

A brief discussion of the Discrete-Invariant Imbedding Algorithm also called Thomas Algorithm is presented as follows:

Consider the relation

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{36}$$

$$\mathrm{i} = 1, 2, ..., \mathrm{N} - 1;$$

where E_i, F_i, G_i and H_i are known and

$$y_0 = y(0) = \alpha \tag{37a}$$

$$y_N = y(1) = \beta \tag{37b}$$

Consider a difference relation of the form

$$y_i = W_i y_{i+1} + T_i, \quad \mathrm{i} = \mathrm{N} - 1, \mathrm{N} - 2, ..., 2, 1 \tag{38}$$

where W_i and T_i corresponding to $W(x_i)$ and $T(x_i)$ are to be determined.

From (38) we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{39}$$

Substituting (39) in (36), we get

$$E_i (W_{i-1} y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i$$

$$y_i = \frac{G_i}{F_i - E_i W_{i-1}} y_{i+1} + \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \quad (40)$$

By comparing (40) and (38) we get the recurrence relations

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}} \quad (41)$$

and

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \quad (42)$$

To solve these recurrence relations for $i = 1,2,\ldots N-1$, we need to know the initial conditions for W_0 and T_0. This can be done by considering (37a)

$$y_0 = \alpha = W_0 y_1 + T_0$$

If we choose W_0=0, then $T_0 = \alpha$. With these initial values, we compute sequentially W_i and T_i for $i = 1,2,\ldots N-1$ from (41) and (42) in the forward process and then obtain y_i in the backward process from (38) using (37a).

For further discussion on the stability of Thomas Algorithm one can refer (Angel and Bellman [1], Elsgolt's and Norkin [5] and Kadalbajoo and Reddy [6]).

Here under the assumptions $a(x) > 0$, $\mathrm{b(x)} < 0$ and $(\varepsilon - \delta a(x)) > 0$, the diagonal dominance property $F_i \geq E_i + G_i$ and $|E_i| \leq |G_i|$ $E_i > 0$, $G_i > 0$, holds true and thus Thomas algorithm is stable.

4. Numerical Experiments

To illustrate the applicability of the method, two numerical examples with both left-end and right-end boundary layer are considered. The computed results are compared with exact solution of the problems and on the problems whose exact solutions are not known, for different values of δ of $\mathrm{o}(\varepsilon)$.

Example 1. Consider the singularly perturbed differential difference equation

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0; \quad x \in [0,1] \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y(1) = 1.$$

The exact solution to this problem is given by

$$y(x)=\frac{\left(1-e^{m_2}\right)e^{m_1x}+\left(e^{m_1}-1\right)e^{m_2x}}{e^{m_1}-e^{m_2}}$$

where $m_1=\dfrac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$ and $m_2=\dfrac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$

Computational results are presented in the tables 1, 2, 3 and 4 for $\varepsilon=0.01$ and 0.001 for different values of δ.

Example 2. Consider an example with variable coefficient singularly perturbed differential difference equation with left layer:

$\varepsilon y''(x)+e^{-0.5x}y'(x-\delta)-y(x)=0$ with $y(0)=1,\ y(1)=1$

The exact solution is not known for this problem. The effect of δ on the left boundary layer is shown with computational results, for its different values in tables 5 and 6.

Example 3. Consider the singularly perturbed differential difference equation

$$\varepsilon y''(x)-y'(x-\delta)-y(x)=0;\ x\in[0,1] \text{ with } y(0)=1 \text{ and } y(1)=-1.$$

The exact solution is given by $y(x)=\dfrac{\left(1+e^{m_2}\right)e^{m_1x}-\left(e^{m_1}+1\right)e^{m_2x}}{e^{m}-e^{m_1}}$

where $m_1=\dfrac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$ and $m_2=\dfrac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}$

Computational results are presented in the tables 7, 8 and 9 for $\varepsilon=0.01$ and 0.001 for different values of δ.

Example 4. Consider an example with variable coefficient singularly perturbed differential difference equation with right layer:

$\varepsilon y''(x)-e^{x}y'(x-\delta)-y(x)=0$ with $y(0)=1,\ y(1)=1$

The exact solution is not known for this problem. The effect of δ on the left boundary layer is shown with computational results, for its different values in 10 and 11.

Table-1. Numerical results of Example-1 for $\varepsilon = 0.01,\ \ \delta = 0.001,$ N=100

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.03	0.3901927	0.3900410	1.516e-04
0.04	0.3877603	0.3874347	3.256e-04
0.05	0.3901408	0.3897593	3.814e-04
0.07	0.3975241	0.3971217	4.024e-04
0.09	0.4054658	0.4050624	4.033e-04
0.30	0.4993732	0.4989908	3.823e-04
0.50	0.6089609	0.6086279	3.330e-04
0.70	0.7425978	0.7423541	2.437e-04
0.90	0.9055613	0.9054623	9.907e-05
1.00	1.0000000	1.0000000	0.000e+00

Table-2. Numerical results of Example-1 for $\varepsilon = 0.01,\ \ \delta = 0.003,$ N=100

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.04	0.3932775	0.3932546	2.291e-05
0.05	0.3925070	0.3923166	1.903e-04
0.07	0.3983441	0.3980571	2.869e-04
0.09	0.4061028	0.4058020	3.008e-04
0.20	0.4528155	0.4525184	2.971e-04
0.40	0.5520024	0.5517307	2.717e-04
0.60	0.6729157	0.6726949	2.208e-04
0.80	0.8203144	0.8201798	1.346e-04
1.00	1.0000000	1.0000000	0.000e+00

Table-3. Numerical results of example-1 for $\varepsilon = 0.001,\ \ \delta = 0.0003,$ N=100

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.02	0.3771424	0.3753846	1.757e-03
0.03	0.3809138	0.3791565	1.757e-03
0.04	0.3847229	0.3829663	1.756e-03
0.06	0.3924559	0.3907012	1.754e-03
0.08	0.4003442	0.3985923	1.751e-03
0.20	0.4511179	0.4494008	1.717e-03
0.40	0.5504496	0.5488774	1.572e-03
0.60	0.6716531	0.6703736	1.279e-03
0.80	0.8195444	0.8187634	7.809e-04
1.00	1.0000000	1.0000000	0.000e+00

Table-4. Numerical results of example-1 for $\varepsilon = 0.001,\ \ \delta = 0.0008$ N=100

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.02	0.3771424	0.3753846	1.757e-03
0.03	0.3809138	0.3791565	1.757e-03
0.04	0.3847229	0.3829663	1.756e-03
0.06	0.3924559	0.3907012	1.754e-03
0.08	0.4003442	0.3985923	1.751e-03
0.20	0.4511179	0.4494008	1.717e-03
0.40	0.5504496	0.5488774	1.572e-03
0.60	0.6716531	0.6703736	1.279e-03
0.80	0.8195444	0.8187634	7.809e-04
1.00	1.0000000	1.0000000	0.000e+00

Table-5. Numerical results for example-2 with $\varepsilon = 0.01,\ \ N = 100$, different values of δ

x	Numerical Solutions				
	$\delta = 0.000$	δ =0.001	δ =0.002	δ =0.003	δ =0.004
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3837599	0.3622626	0.3402440	0.3182891	0.2980169
0.04	0.3068927	0.3013019	0.2968560	0.2937033	0.2918980
0.06	0.3016131	0.3001263	0.2990281	0.2982371	0.2976445
0.08	0.3062093	0.3055005	0.3048846	0.3043271	0.3038031
0.10	0.3123426	0.3117661	0.3112130	0.3106767	0.3101567
0.20	0.3471893	0.3466320	0.3460832	0.3455471	0.3450310
0.40	0.4360086	0.4354570	0.4349182	0.4343990	0.4339123
0.60	0.5602958	0.5597999	0.5593202	0.5588658	0.5584540
0.80	0.7383421	0.7380000	0.7376729	0.7373696	0.7371067
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

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Table-6. Numerical results of example2 for $\varepsilon = 0.001$, *N=100*, different values of δ

x	Numerical Solutions				
	$\delta = 0.0001$	δ =0.0002	δ =0.0003	δ =0.0004	δ =0.0008
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.2792305	0.2792217	0.2792170	0.2792148	0.2792134
0.03	0.2848682	0.2848681	0.2848681	0.2848681	0.2848681
0.05	0.2906957	0.2906956	0.2906955	0.2906955	0.2906955
0.07	0.2967024	0.2967023	0.2967023	0.2967023	0.2967023
0.09	0.3028952	0.3028951	0.3028951	0.3028951	0.3028951
0.10	0.3060636	0.3060635	0.3060635	0.3060634	0.3060634
0.20	0.3406137	0.3406136	0.3406136	0.3406136	0.3406136
0.40	0.4289314	0.4289313	0.4289313	0.4289312	0.4289312
0.60	0.5533208	0.5533207	0.5533207	0.5533207	0.5533207
0.80	0.7330194	0.7330193	0.7330193	0.7330193	0.7330193
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table-7. Numerical results example-3 for $\varepsilon = 0.01, \delta = 0.002$,N=100

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.20	0.8207411	0.8206521	8.898e-05
0.40	0.6736160	0.6734699	1.460e-04
0.60	0.5528644	0.5526845	1.798e-04
0.80	0.4537585	0.4535617	1.967e-04
0.90	0.4108110	0.4105821	2.289e-04
0.91	0.4064063	0.4061459	2.604e-04
0.93	0.3955821	0.3951281	4.540e-04
0.95	0.3720079	0.3708222	1.185e-03
0.97	0.2774855	0.2740694	3.416e-03
1.00	-1.0000000	-1.0000000	0.000e+00

Table-8: Numerical results for example-3, $\varepsilon = 0.01$, $\delta = 0.003$, N=100

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.000e+00
0.20	0.8208852	0.8208084	7.678e-05
0.40	0.6738526	0.6737265	1.260e-04
0.60	0.5531556	0.5530004	1.552e-04
0.80	0.4540771	0.4539072	1.699e-04
0.91	0.4062358	0.4059562	2.796e-04
0.93	0.3939262	0.3933562	5.700e-04
0.95	0.3650479	0.3635169	1.531e-03
0.97	0.2552733	0.2511992	4.074e-03
0.98	0.0969142	0.0910572	5.857e-03
1.00	-1.0000000	-1.000000	0.000e+00

Table-9: Numerical results for example-3,with $\varepsilon = 0.001$, $\delta = 0.008$

x	Numerical solution	Exact Solution	Absolute Error
0.00	1.0000000	1.0000000	0.00e+00
0.20	0.8195506	0.8190244	0.0005.261e-04
0.40	0.6716632	0.6708011	0.0008.621e-04
0.60	0.5504620	0.5494025	0.001.059e-03
0.80	0.4511315	0.4499741	0.001.157e-03
0.91	0.4043615	0.4031817	0.001.179e-03
0.93	0.3963943	0.3952123	0.001.181e-03
0.95	0.3885841	0.3874005	0.001.183e-03
0.97	0.3809276	0.3797430	0.001.184e-03
1.00	-1.0000000	-1.0000000	0.000e+00

Table-10. Numerical results for example-4 with $\varepsilon = 0.01$, $N = 100$, different values of δ

x	Numerical Solutions			
	$\delta = 0.001$	δ =0.003	δ =0.006	δ =0.008
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8371138	0.8375826	0.8383120	0.8387939
0.40	0.7232342	0.7239635	0.7250829	0.7258210
0.60	0.6413734	0.6422490	0.6435791	0.6444547
0.80	0.5811565	0.5821147	0.5835583	0.5845075
0.90	0.5570088	0.5579934	0.5594779	0.5604928
0.91	0.5547757	0.5557627	0.5572619	0.5583385
0.93	0.5504018	0.5513940	0.5530442	0.5545947
0.95	0.5461472	0.5471815	0.5501803	0.5541782
0.97	0.5420377	0.5445834	0.5593688	0.5740964
0.99	0.5859178	0.6073731	0.6865318	0.7232474
1.00	-1.0000000	1.0000000	1.0000000	1.0000000

Table-11. Numerical results for example-4 with $\varepsilon = 0.001$, $N = 100$, different values of δ

x	Numerical Solutions			
	$\delta = 0.0001$	δ =0.0003	δ =0.0006	δ =0.0008
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.20	0.8356445	0.8356568	0.8357367	0.8358666
0.40	0.7213170	0.7213359	0.7214591	0.7216600
0.60	0.6394080	0.6394306	0.6395776	0.6398177
0.80	0.5792895	0.5793140	0.5794742	0.5797360
0.90	0.5552104	0.5552355	0.5553998	0.5556685
0.91	0.5529845	0.5530097	0.5531744	0.5534436
0.93	0.5486249	0.5486502	0.5488155	0.5490859
0.95	0.5443848	0.5444102	0.5445762	0.5448496
0.97	0.5402603	0.5402859	0.5404757	0.5410067
0.99	0.5364624	0.5388005	0.5536257	0.5761217
1.00	1.0000000	1.0000000	1.0000000	1.0000000

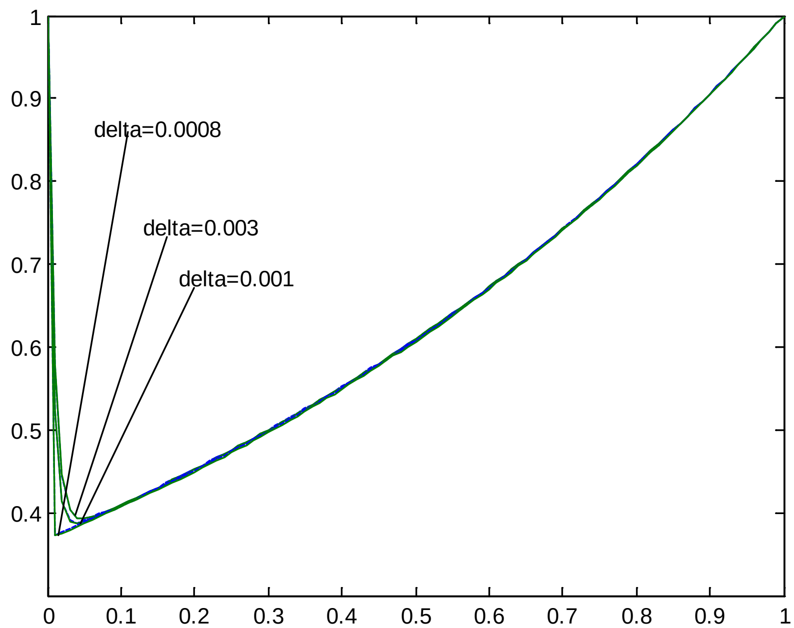

Figure 1: Graph of numerical solution of Example 1 for different values of δ.

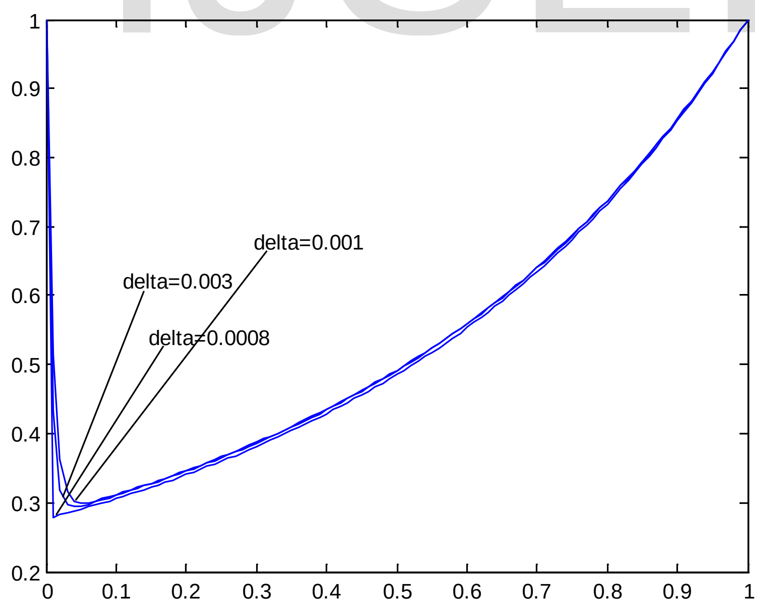

Figure 2: Graph of numerical solution of Example 2 for different values of δ.

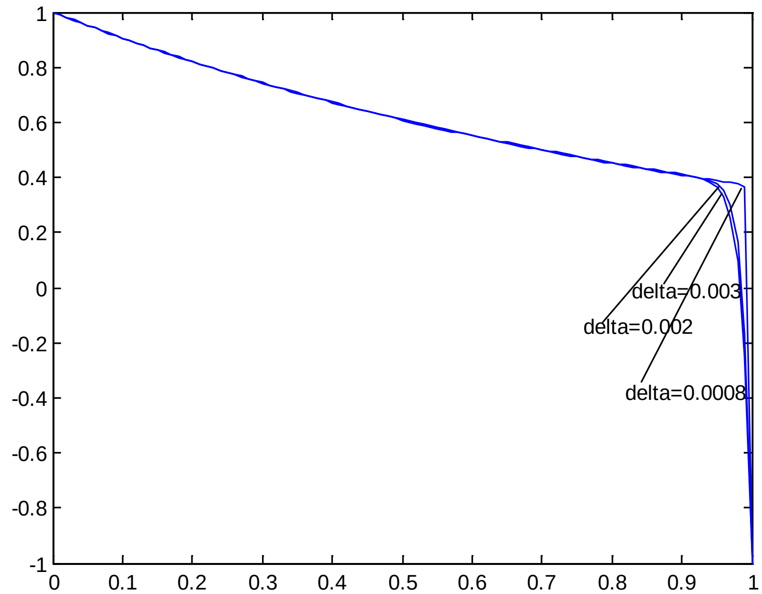

Figure 3: Graph of numerical solution of Example 3 for different values of δ.

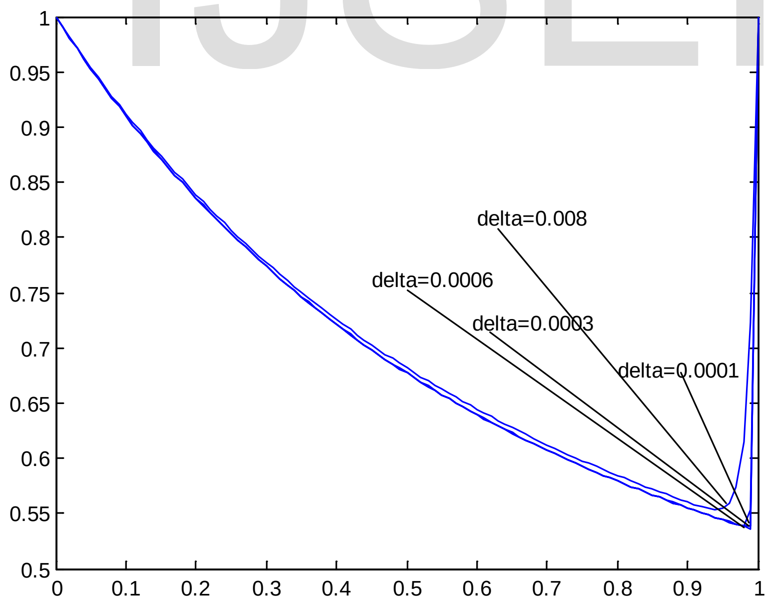

Figure 4: Graph of numerical solution of Example 4 for different values of δ.

Discussion and Conclusion

We have presented fitted modified upwind scheme to solve singularly perturbed differential equations. The scheme is repeated for different choices of the delay parameter, δ and perturbation parameter ε. The choice of δ is not unique, but can assume any number of values satisfying the condition, $0<\delta<<1$. The mesh size h is fixed so as to reduce the amount of computations, and the value of δ varies. The solutions are calculated for all the values of h but only few values have been reported. The numerical results tabulated in the tables 1-11 are in the support of the theory. The graphs of the solution for the considered examples are plotted for different values of δ.

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